

The Schwinger Dyson equations and the algebra of constraints of random tensor models at all orders

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Random tensor models for a generic complex tensor generalize matrix models in arbitrary dimensions and yield a theory of random geometries. They support a $1/N$ expansion dominated by graphs of spherical topology. Their Schwinger Dyson equations, generalizing the loop equations of matrix models, translate into constraints satisfied by the partition function. The constraints have been shown, in the large N limit, to close a Lie algebra indexed by colored rooted D -ary trees yielding a first generalization of the Virasoro algebra in arbitrary dimensions. In this paper we complete the Schwinger Dyson equations and the associated algebra at all orders in $1/N$. The full algebra of constraints is indexed by D -colored graphs, and the leading order D -ary tree algebra is a Lie subalgebra of the full constraints algebra.

Keywords: Random tensor models, $1/N$ expansion, critical behavior

I. INTRODUCTION

Random matrix models encode a theory of random Riemann surfaces dual to the ribbon Feynman graphs generated by their perturbative expansion [1]. The amplitudes of the Feynman graphs depend on the size of the matrix, N , and the perturbative series can be reorganized in powers of $1/N$ [2]. At leading order only planar graphs contribute [3]. The planar family is summable and undergoes a phase transition to continuous Riemann surfaces [4, 5] when the coupling constant approaches some critical value. Single and multi-matrix models have been very successful in describing the critical behavior of two-dimensional statistical models on random geometries [6–9] and via the KPZ correspondence [10–13] on fixed geometries. Subleading terms in the $1/N$ expansion can be accessed through double scaling limits [14–16].

Random matrices are analyzed with many techniques, like gauge fixing to the eigenvalues of the matrix, orthogonal polynomials and so on [1]. Of particular interest for this paper are the Schwinger Dyson equations (SDE) (or loop equation) [17–20] which translate into a set of Virasoro constraints satisfied by the partition function.

The success of matrix models in describing random two dimensional surfaces inspired their generalization in higher dimensions to random tensor models [21–23] (see also [24–30] for more recent developments). The corresponding theory of random higher dimensional geometries was initially hoped to give insights into conformal field theory, statistical models in random geometry and quantum gravity in three and four dimensions. In spite of these initial high hopes, tensor models have for a long time been unsuccessful in providing an analytically controlled theory of random geometries: until recently all the nice aspects of matrix models could not be generalized to tensors, as their $1/N$ expansion was missing.

The situation has changed with the discovery of *colored* [31–33] rank $D \geq 3$ random tensor models¹. Their perturbation series supports a $1/N$ expansion [35–37], indexed by the *degree*, a positive integer which plays in higher dimensions the role of the genus. We emphasize that the degree is *not* a topological invariant. Leading order graphs, baptized *melonic* [38], triangulate the D -dimensional sphere in any dimension [35, 36]. They form a summable family as they map to colored rooted D -ary trees [38]. Colored random tensors [39] gave the first analytically accessible theory of random geometries in three and more dimensions and became a rapidly expanding field [40–46], (see also [47] for some related developments). In particular the first applications of random tensors to statistical models in random geometries have been explored [48–51].

The results obtained for the colored models have been shown to hold in fact for tensor models for a single, complex, generic (that is non symmetric) tensor [52]. The colors become a canonical bookkeeping device tracking the indices of the tensor. The theory thus obtained is universal [53] and constitutes the only analytically controlled generalization of matrix models to higher dimensions we have so far.

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¹ In $D = 2$ colored matrix models have been studied [34], but the colors do not play the key role they play in three and more dimensions.

A posteriori one understands why the discovery of the $1/N$ expansion for tensor models has taken so long: in order to have any control on the amplitude of graphs one needs to be able to track the indices of the tensors (this is achieved by the colors). This is impossible if one uses tensors with symmetry properties under permutations of the indices. Tensor models for generic tensors have been generalized to tensor field theories, [54–56] in which the quadratic part is not invariant under the $\otimes^D U(N)$ symmetry. The non trivial quadratic part generates a renormalization group flow and the $1/N$ expansion is recovered dynamically. We already possess an example of a tensor field theory which is asymptotically free [55]. Due to the universality [53] of tensor measures one is tempted to conjecture that asymptotic freedom will be a feature of all tensor field theories.

In order to study this universal theory of random tensors one must generalize to higher dimensions as many of the tools which were so useful in matrix models as possible. This turns out again to be a non trivial problem. For instance, although various generalizations of eigenvalues to tensors have been proposed, they lack most of the interesting properties of the eigenvalues of matrices (for one the tensor does not diagonalize under a change of basis). A generalization of the notion of determinant to tensors (formats), the Gel’fand hyperdeterminant [57] exist (and interestingly enough it requires that the tensor indices are distinguished), however one does not possess an explicit formula for it (except for the (1,1,1) format, in which case one gets just the well known Cayley hyperdeterminant). However, 2 is a rather small value for a supposedly large N .

The only set of techniques which can be generalized relatively straightforwardly to tensors are the SDE’s. At leading order the SDE’s of tensor models have been derived in [58]. As one writes a SDE for every observable of the model, the equations are indexed by the observables. At leading order only melonic observables contribute and, as they map to trees, the SDE’s and associated Lie algebra of constraints are indexed by trees. The definition of the algebra relies on a gluing operation for trees, $\mathcal{T}_1 \star_V \mathcal{T}_2$, which generalizes the addition of integers (in terms of composition of observables for matrix models the addition encodes the gluing of two cycles of lengths p and q) to an “addition” of observables indexed by trees. Thus the Virasoro (strictu sensu the positive part of the Witt) algebra of constraints of matrix models becomes

$$[L_m, L_n] = (m - n)L_{m+n} \Rightarrow [\mathcal{L}_{\mathcal{T}_1}, \mathcal{L}_{\mathcal{T}_2}] = \sum_{V \in \mathcal{T}_1} \mathcal{L}_{\mathcal{T}_1 \star_V \mathcal{T}_2} - \sum_{V \in \mathcal{T}_2} \mathcal{L}_{\mathcal{T}_2 \star_V \mathcal{T}_1} . \quad (1.1)$$

In [58] it is proved that the constraints hold at leading order in N , that is $\lim_{N \rightarrow \infty} N^{-D} \frac{1}{Z} \mathcal{L}_{\mathcal{T}} Z = 0$, with Z the partition function of tensor models. In this paper we complete the algebra of constraints to all order in $1/N$. This time the observables (hence the SDE’s and the algebra of constraints) are indexed by D -colored graphs \mathcal{B} with a distinguished vertex \bar{v} . The observables can be glued by a $\star_{(v, \bar{v}')}$ operation and we obtain a Lie algebra of constraints

$$\mathcal{L}_{(\mathcal{B}_1, \bar{v}_1)} Z = 0 , \quad \left[\mathcal{L}_{(\mathcal{B}_1, \bar{v}_1)}, \mathcal{L}_{(\mathcal{B}_2, \bar{v}_2)} \right] = \sum_{v \in \mathcal{B}_1} \mathcal{L}_{(\mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2, \bar{v}_1)} - \sum_{v \in \mathcal{B}_2} \mathcal{L}_{(\mathcal{B}_2 \star_{(v, \bar{v}_1)} \mathcal{B}_1, \bar{v}_2)} , \quad (1.2)$$

obeyed at all orders in N . When restricted to melonic observables the $\star_{(v, \bar{v})}$ composition of observables particularizes to the \star_V composition of the associated trees. The melonic observables are closed under this gluing. We thus not only dispose of the full set of constraints for random tensor models, but, non trivially, the D -ary tree subalgebra closes into a Lie subalgebra of the full constraints algebra *at all* orders in $1/N$.

The algebra of constraints is the starting point of the study of the continuum limit [52] of tensor models. In matrix models the continuum theory, Liouville gravity coupled with various matter fields at a conformal point, is identified using the theory of (unitary and non unitary) representations of the Virasoro algebra. The continuum operators, which are composite operators in terms of the loop observables, are hard to identify [20]. One uses the theory of representations of the Virasoro algebra to infer the appropriate correspondence. The same must be done in higher dimensions. We need first to study and classify the central extensions and unitary representations of this algebra. Once the field content of the continuum theory is identified, we need to translate the continuum observables in terms of the observables of tensor models. This study will provide a landscape of universality classes of continuum theories of random geometries accessible using tensor models in arbitrary dimensions. The algebra of constraints identified in [58] and completed in this paper at all order constitutes the first step towards applying tensor models to the study of conformal field theories and quantum gravity in arbitrary dimensions.

This paper is organized as follows. In section II we recall the generic one tensor models and their $1/N$ expansion. In section III we define two graphical operations on arbitrary observables of such models, derive the SDE’s and prove that they close a Lie algebra.

II. THE $1/N$ EXPANSION OF TENSOR MODELS

A. Tensor invariants and action

Let H_1, \dots, H_D be complex vector spaces of dimensions N_1, \dots, N_D . A covariant tensor $T_{n_1 \dots n_D}$ of rank D (or a $(N_1 - 1, \dots, N_D - 1)$ format) is a collection of $\prod_{i=1}^D N_i$ complex numbers supplemented with the requirement of covariance under base change. We consider tensors T transforming under the *external* tensor product of fundamental representations of the unitary group $\otimes_{i=1}^D U(N_i)$, that is each $U(N_i)$ acts independently on its corresponding H_i . The complex conjugate tensor $\bar{T}_{n_1 \dots n_D}$ is then a rank D contravariant tensor. They transform as

$$T'_{a_1 \dots a_D} = \sum_{n_1, \dots, n_D} U_{a_1 n_1} \dots V_{a_D n_D} T_{n_1 \dots n_D}, \quad \bar{T}'_{a_1 \dots a_D} = \sum_{n_1, \dots, n_D} \bar{U}_{a_D n_D} \dots \bar{V}_{a_1 n_1} \bar{T}_{n_1 \dots n_D}. \quad (2.3)$$

We will denote the indices of the complex conjugated tensor with a bar, and use the shorthand notation $\bar{n} = (n_1, \dots, n_D)$. We restrict from now on to $N_i = N, \forall i$.

Among the invariants built out of T and \bar{T} we will deal in the sequel exclusively with *trace invariants*. They are obtained by contracting two by two covariant with contravariant indices in a polynomial in the tensor entries,

$$\text{Tr}(T, \bar{T}) = \sum \prod \delta_{n_i, \bar{n}_i} T_{n_1 \dots n_D} \dots \bar{T}_{\bar{n}_1 \dots \bar{n}_D}, \quad (2.4)$$

where *all* indices are saturated. A trace invariant has the same number of T and \bar{T} . As T_{n_1, \dots, n_D} transforms as a complex vector under the action of the unitary group on only one index, one can use the fundamental theorem of classical invariants for $U(N)$ (whose origins can be traced back to Gordan [59], see [60] and references therein) successively for each index and conclude that *all* invariant polynomials in the tensor entries write as linear combination of trace invariants.

Trace invariants are labeled by graphs. To draw the graph associated to a trace invariant we represent every T by a white vertex v and every \bar{T} by a black vertex \bar{v} . We promote the position of an index to a color: n_1 has color 1, n_2 has color 2 and so on. The contraction of two indices n_i and \bar{n}_i of two tensors is represented by a line $l^i = (v, \bar{v})$ connecting the corresponding two vertices. Lines inherit the color of the index, and always connect a black and a white vertex. Any trace invariant is thus represented by a closed D -colored graph \mathcal{B} [39]. The graph \mathcal{B} is bipartite (it has black and white vertices denoted \bar{v} and v , such that all lines connect a black and a white vertex) and its lines $l^i = (v, \bar{v})$ have a color $i = 1, 2, \dots, D$. All the vertices of \mathcal{B} have coordination D with all the edges incident to a given vertex having distinct colors. Some trace invariants for rank 3 tensors are represented in figure 1.

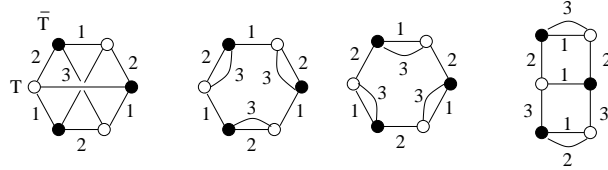


Figure 1. Graphical representation of trace invariants.

The trace invariant associated to the graph \mathcal{B} writes as

$$\text{Tr}_{\mathcal{B}}(T, \bar{T}) = \sum_{\{\bar{n}^v, \bar{n}^{\bar{v}}\}_{v, \bar{v} \in \mathcal{V}}} \delta_{\{\bar{n}^v, \bar{n}^{\bar{v}}\}}^{\mathcal{B}} \prod_{v, \bar{v} \in \mathcal{B}} T_{\bar{n}^v} \bar{T}_{\bar{n}^{\bar{v}}}, \quad \text{with} \quad \delta_{\{\bar{n}^v, \bar{n}^{\bar{v}}\}}^{\mathcal{B}} = \prod_{i=1}^D \prod_{l^i = (v, \bar{v}) \in \mathcal{B}} \delta_{n_i^v \bar{n}_i^{\bar{v}}}. \quad (2.5)$$

where l^i runs over all the lines of color i of \mathcal{B} . The product of Kronecker delta's encoding the index contractions of the observable associated to the graph \mathcal{B} , $\delta_{\{\bar{n}^v, \bar{n}^{\bar{v}}\}}^{\mathcal{B}}$ is called the *trace invariant operator with associated graph \mathcal{B}* [58].

Trace invariant operators factor over the connected components of the graph. We denote $\Gamma_{2k}^{(D)}$ the set of D -colored, connected graphs with $2k$ unlabeled vertices.

Formally, a D colored graph with labeled vertices is defined by an incidence matrix $\epsilon_{v\bar{v}}$ whose entries are $\{i_1, \dots, i_k\}$ if the vertices v and \bar{v} are connected by lines of colors i_1, \dots, i_k . In particular $\epsilon_{v\bar{v}} = \emptyset$ if v and \bar{v} are not connected by any line. An element $\mathcal{B} \in \Gamma_{2k}^{(D)}$ is an equivalence class of incidence matrices related to one another by permutations of lines and columns (corresponding to relabellings of the vertices). One can write the trace invariant associated to \mathcal{B} directly in terms of the incidence matrix of any representative graph with labeled vertices $\prod_{v, \bar{v}} \prod_{i \in \epsilon_{v\bar{v}}} \delta_{n_i^v \bar{n}_i^{\bar{v}}}$.

The subgraphs with two colors of a D -colored graph \mathcal{B} are called *faces*. We denote the number of faces of a graph \mathcal{B} by $F_{\mathcal{B}}$. They will play an important role in the next section. The graphs with $D = 3$ colors represented in figure 1 posses three type of faces, given by the subgraphs with lines of colors $(1, 2)$, $(1, 3)$ and $(2, 3)$.

To every graph \mathcal{B} we can associate a non negative integer, its *degree* $\omega(\mathcal{B})$ [36, 37, 39]. The main property of the degree is that it provides a counting of the number of faces $F_{\mathcal{B}}$: for a closed, connected graph with D colors and $2p_{\mathcal{B}}$ vertices the total number of faces is [37, 39, 52]

$$F_{\mathcal{B}} = \frac{(D-1)(D-2)}{2} p_{\mathcal{B}} + (D-1) - \frac{2}{(D-2)!} \omega(\mathcal{B}) . \quad (2.6)$$

The degree provides in higher dimensions a generalization of the genus of ribbon graphs, and indexes their the $1/N$ expansion. It is however *not* a topological invariant, but it combines topological and combinatorial information about the graph [43]. Of course the degree can be defined for graphs with $D+1$ colors (say $0, 1, \dots, D$), and the number of faces of graph with $D+1$ colors is $F_{\mathcal{G}} = \frac{D(D-1)}{2} p_{\mathcal{G}} + D - \frac{2}{(D-1)!} \omega(\mathcal{G})$. Another important property of the degree (see for instance [39]) is the following. Consider a closed connected graph \mathcal{G} with $D+1$ colors $0, 1, \dots, D$, and denote \mathcal{B}_{ρ} its connected subgraphs of colors $1, \dots, D$ (where $\rho = 1 \dots |\rho|$ labels the connected components). Then

$$\omega(\mathcal{G}) \geq D \sum_{\rho=1}^{|\rho|} \omega(\mathcal{B}_{\rho}) . \quad (2.7)$$

Going back to invariants one can build out of a complex tensor, we note that there exists a unique D -colored graph with two vertices, namely the graph in which all the lines connect the two vertices. We call it the D -dipole and its associated invariant is

$$\text{Tr}_{\text{dipole}}(T, \bar{T}) = \sum_{\vec{n}, \vec{\bar{n}}} T_{\vec{n}} \bar{T}_{\vec{\bar{n}}} \left[\prod_{i=1}^D \delta_{n_i \bar{n}_i} \right] . \quad (2.8)$$

The most general single trace invariant action for a non-symmetric tensor is

$$S(T, \bar{T}) = t_1 \text{Tr}_{\text{dipole}}(T, \bar{T}) + \sum_{k=2}^{\infty} \sum_{\mathcal{B} \in \Gamma_{2k}^{(D)}} t_{\mathcal{B}} \text{Tr}_{\mathcal{B}}(T, \bar{T}) , \quad (2.9)$$

where $t_{\mathcal{B}}$ are the coupling constants associated to the D -colored graphs \mathcal{B} and we singled out the quadratic part corresponding to the D -dipole. In equation (2.9) one sometimes adds a scaling factor $N^{-\frac{2}{(D-2)!} \omega(\mathcal{B})}$ for every trace invariant (as $\omega(\mathcal{B}) \geq 0$ this suppresses some of them). Adding this extra scaling would simplify some formulae but does not modify anything in the sequel. We will treat in this paper the most general single trace rank D tensor model defined by the partition function

$$Z(t_{\mathcal{B}}) = \exp(-F(t_{\mathcal{B}})) = \int d\bar{T} dT \exp \left(-N^{D-1} S(T, \bar{T}) \right) . \quad (2.10)$$

B. Graph amplitudes

The invariant observables are the trace invariants represented by D -colored graphs. The Feynman graphs contributing to the expectation of an observable are obtained by Taylor expanding with respect to $t_{\mathcal{B}}$ and evaluating the Gaussian integral in terms of Wick contractions. The D -colored graphs \mathcal{B} associated to the invariants $\text{Tr}_{\mathcal{B}}(T, \bar{T})$ in the action act as *effective Feynman vertices*. The effective vertices are connected by effective *propagators* (Wick contractions, pairings of T 's and \bar{T} 's). A Wick contraction of two tensor entries $T_{a_1 \dots a_D}$ and $\bar{T}_{\bar{p}_1 \dots \bar{p}_D}$ with the quadratic part (2.8) consists in replacing them by $\frac{1}{N^{D-1} t_1} \prod_{i=1}^D \delta_{a_i \bar{p}_i}$. We will represent the Wick contractions as dashed lines labeled by the fictitious color 0. The dashed lines of color 0 are thus very different from the solid lines of colors $1, 2, \dots, D$: they identify *all* the indices of the two vertices (one white corresponding to T and one black corresponding to \bar{T}) it connects (recall that the lines of colors $1, 2, \dots, D$ identify only one index each). An example of a Feynman graph is presented in figure 2.

The Feynman graphs \mathcal{G} are therefore $(D+1)$ -colored graphs. We will keep the notation \mathcal{B} for the D -colored graphs and denote \mathcal{G} the $(D+1)$ -colored graph. We call the connected components with colors $1, \dots, D$ of \mathcal{G} , denoted \mathcal{B}_{ρ} with $\rho = 1, \dots, |\rho|$, the D -bubbles of \mathcal{G} . For instance the 3-bubbles with colors 123 of the graph in figure 2 are the subgraphs made of solid lines.

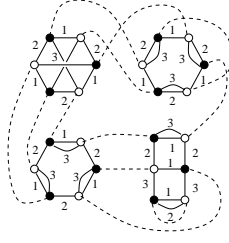


Figure 2. A Feynman graph of tensor models.

A graph \mathcal{G} has two kinds of faces: those with colors $i, j = 1, \dots, D$ (which belong also to some D -bubble \mathcal{B}), and those with colors $0, i$, for $i = 1, \dots, D$. We denoted $F_{\mathcal{G}}^{0i}$ the number of faces of colors $0i$ of \mathcal{G} .

The free energy has an expansion in closed, connected $(D+1)$ -colored graphs,

$$F(t_{\mathcal{B}}) = \sum_{\mathcal{G} \in \Gamma^{(D+1)}} \frac{(-1)^{|\rho|}}{s(\mathcal{G})} A(\mathcal{G}), \quad (2.11)$$

where $s(\mathcal{G})$ is a symmetry factor and $|\rho|$ is the number of effective vertices (that is subgraphs with colors $1 \dots D$ or D -bubbles). The amplitudes of \mathcal{G} is

$$A(\mathcal{G}) = \prod_{\rho} t_{\mathcal{B}_{\rho}} \sum_{\{\vec{n}^v, \vec{n}^{\bar{v}}\}} \left[\prod_{\rho} N^{D-1} \delta_{\{\vec{n}^v, \vec{n}^{\bar{v}}\}}^{\mathcal{B}_{\rho}} \right] \left[\prod_{l^0=(v, \bar{v}) \in \mathcal{G}} \frac{1}{t_1 N^{D-1}} \prod_i \delta_{n_i^v, \bar{n}_i^{\bar{v}}} \right]. \quad (2.12)$$

An index n_i is identified along the lines of color i in \mathcal{B}_{ρ} and along the dashed lines of color 0. We thus obtain a free sum per face of colors $0i$, so that

$$A(\mathcal{G}) = \frac{\prod_{\rho} t_{\mathcal{B}_{(\rho)}}}{t_1^{|l^0|}} N^{(D-1)|\rho| - (D-1)|l^0| + \sum_i |F_{\mathcal{G}}^{0i}|}, \quad (2.13)$$

where $|l^0|$ denotes the total number of lines of color 0 of \mathcal{G} . But $\sum_i F_{\mathcal{G}}^{0i} = F_{\mathcal{G}} - \sum_{\rho} F_{\rho}$, where $F_{\mathcal{G}}$ denotes the total number of faces of \mathcal{G} and F_{ρ} the number of faces of the D -bubble \mathcal{B}_{ρ} . Using (2.6) for each \mathcal{B}_{ρ} and for \mathcal{G} (taking into account that \mathcal{G} has $D+1$ colors) and noting that $|l^0| = p$, with p the half number of vertices of \mathcal{G} , we obtain

$$\begin{aligned} A(\mathcal{G}) &= \frac{\prod_{\rho} t_{\mathcal{B}_{(\rho)}}}{t_1^p} N^{D - \frac{2}{(D-1)!} \omega(\mathcal{G}) + \frac{2}{(D-2)!} \sum_{\rho=1}^{|\rho|} \omega(\mathcal{B}_{\rho})} \\ &= \frac{\prod_{\rho} t_{\mathcal{B}_{(\rho)}}}{t_1^p} N^{D - \frac{2}{D!} \omega(\mathcal{G}) - \frac{2}{D(D-2)!} \left(\omega(\mathcal{G}) - D \sum_{\rho=1}^{|\rho|} \omega(\mathcal{B}_{\rho}) \right)}, \end{aligned} \quad (2.14)$$

with $\omega(\mathcal{G})$ the degree of the graph \mathcal{G} and $\omega(\mathcal{B}_{\rho})$ the degree of \mathcal{B}_{ρ} . The amplitude of a graph \mathcal{G} is thus at most N^D and it is suppressed with the degree $\omega(\mathcal{G})$. This is the $1/N$ expansion for random tensor models [37, 52, 58]. Expectation values of the observables have similar expansions in $1/N$

$$\frac{\left\langle \text{Tr}_{\mathcal{B}_1}(T, \bar{T}) \dots \text{Tr}_{\mathcal{B}_{|\alpha|}}(T, \bar{T}) \right\rangle_c}{Z} = \sum_{\mathcal{G} \supset \cup_{\alpha} \mathcal{B}_{\alpha}} N^{D - |\alpha|(D-1) - \frac{2}{(D-1)!} \omega(\mathcal{G}) + \frac{2}{(D-2)!} \sum_{\rho=1}^{|\rho|} \omega(\mathcal{B}_{\rho})} \frac{\prod_{\rho \neq \alpha} t_{\mathcal{B}_{\rho}}}{t_1^p}. \quad (2.15)$$

where the sum runs over all $D+1$ colored graphs \mathcal{G} with $|\alpha|$ marked D -bubbles \mathcal{B}_{α} , and \mathcal{B}_{ρ} denotes all the D -bubbles of \mathcal{G} (hence the α 's are some of the ρ 's).

III. SCHWINGER DYSON EQUATIONS

The Schwinger Dyson equations of the model write in terms of two graphical operations. They encode the changes an observable \mathcal{B} undergoes when adding a line of color 0. The D -colored graphs \mathcal{B} represent $D-1$ dimensional closed connected pseudomanifolds [33]. When seen as subgraphs of some $D+1$ colored graph \mathcal{G} they become chunks of a

D dimensional space. The chunk associated to \mathcal{B} is the topological cone over the pseudomanifold [52]. The same holds for matrix models: the invariants $\text{Tr}[(MM^\dagger)^q]$ are either seen as loop observables, or as polygons when acting as effective Feynman vertices. The polygon represented by $\text{Tr}[(MM^\dagger)^q]$ when it appears as an effective vertex in a Feynman graph is the cone over the loop.

The usual loop equations of matrix models encode the change of the loop observables by the addition of a line: a loop can either merge with another loop or it can split into two loops. In the dual triangulation this is seen as the gluing of two segments bounding loops. Either the two segments belong to two distinct loops, and then one obtains a larger loop, or they belong to the same loop, in which case the loop splits into two. A loop observable $\text{Tr}[(MM^\dagger)^{q+1}]$ is fully characterized by its “length” q . The merging of two loops of length p and q leads to a loop with length $p+q$: the merging is just a graphical representation of the addition of integers. The SDE’s of the model are indexed by loops, hence by integers. They can be transformed into a set of constraints on the partition function $L_p Z = 0$. The latter respect the Virasoro algebra [17–20], $[L_p, L_q] = (p-q)L_{p+q}$. Graphically this equation is understood as follows. Given two loops of length p and q , in how many ways can I compose them? I can either glue the loop of length q on the loop of length p (and one has $p+1$ choices for where to insert the second loop), or I can glue the loop of length p on the loop of length q (and one has $q+1$ choices for where to perform this insertion) In all cases I obtain a loop of length $p+q$, associated to the operator L_{p+q} .

A similar picture holds, with the appropriate generalizations, in higher dimensions.

A. Graph operations

The colored gluing of two graphs. Consider two D colored graphs \mathcal{B}_1 and \mathcal{B}_2 and chose two vertices $v_1 \in \mathcal{B}_1$ and $\bar{v}_2 \in \mathcal{B}_2$. We define the colored gluing of graphs at v_1 and \bar{v}_2 as the graph $\mathcal{B}_1 \star_{(v_1, \bar{v}_2)} \mathcal{B}_2$ obtained by deleting the vertices v_1 and \bar{v}_2 and joining all the lines touching them pairwise respecting the colorings.

This operation can be performed in two steps. Consider the trace invariants associated to \mathcal{B}_1 and \mathcal{B}_2 and connect the two vertices v_1 and \bar{v}_2 by a dashed line of color 0. Forgetting for a second the scalings with N , this line will identify all the indices of the tensor associated to v_1 with the ones of the tensor associated to \bar{v}_2

$$\begin{aligned} \sum_{n_i^{v_1}, \bar{n}_i^{\bar{v}_2}} \delta_{\{\bar{n}^v, \bar{n}^{\bar{v}}\}}^{\mathcal{B}_1} \left(\prod_i \delta_{n_i^{v_1} \bar{n}_i^{\bar{v}_2}} \right) \delta_{\{\bar{n}^v, \bar{n}^{\bar{v}}\}}^{\mathcal{B}_2} &= \sum_{n_i^{v_1}, \bar{n}_i^{\bar{v}_2}} \left(\prod_{i=1}^D \prod_{l^i=(v, \bar{v}) \in \mathcal{B}_1} \delta_{n_i^v \bar{n}_i^{\bar{v}}} \right) \left(\prod_i \delta_{n_i^{v_1} \bar{n}_i^{\bar{v}_2}} \right) \left(\prod_{i=1}^D \prod_{l^i=(v, \bar{v}) \in \mathcal{B}_2} \delta_{n_i^v \bar{n}_i^{\bar{v}}} \right) \\ &= \delta_{\{\bar{n}^v, \bar{n}^{\bar{v}}\}}^{\mathcal{B}_1 \star_{(v_1, \bar{v}_2)} \mathcal{B}_2}, \end{aligned} \quad (3.16)$$

as v_1 is a vertex in \mathcal{B}_1 and \bar{v}_2 is a vertex in \mathcal{B}_2 . The gluing is represented graphically in figure 3.

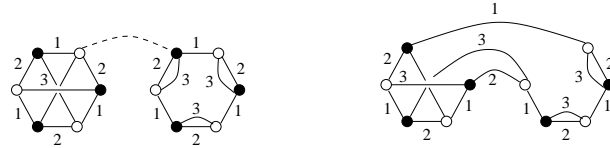


Figure 3. Graphical representation of the colored gluing of two graphs.

Of course the gluing preserves the colorability, hence $\mathcal{B}_1 \star_{(v_1, \bar{v}_2)} \mathcal{B}_2$ is a connected D -colored graph with colors $1, \dots, D$. At the level of the incidence matrix, one builds from $\epsilon_{v\bar{v}}^{\mathcal{B}_1}$ and $\epsilon_{v\bar{v}}^{\mathcal{B}_2}$ the incidence matrix

$$v \neq v_1, \bar{v} \neq \bar{v}_2 \quad \epsilon_{v\bar{v}}^{\mathcal{B}_1 \star_{(v_1, \bar{v}_2)} \mathcal{B}_2} = \begin{cases} \epsilon_{v\bar{v}}^{\mathcal{B}_1} & \text{if } v, \bar{v} \in \mathcal{B}_1 \\ \epsilon_{v\bar{v}}^{\mathcal{B}_2} & \text{if } v, \bar{v} \in \mathcal{B}_2 \\ \epsilon_{v_1 \bar{v}}^{\mathcal{B}_1} \cap \epsilon_{v \bar{v}_2}^{\mathcal{B}_2} & \text{if } v \in \mathcal{B}_2, \bar{v} \in \mathcal{B}_1 \end{cases} = \epsilon_{v\bar{v}}^{\mathcal{B}_1} \cup \epsilon_{v\bar{v}}^{\mathcal{B}_2} \cup \left(\epsilon_{v_1 \bar{v}}^{\mathcal{B}_1} \cap \epsilon_{v \bar{v}_2}^{\mathcal{B}_2} \right). \quad (3.17)$$

The colored contraction of a graph. The second operation is similar to the gluing, but it pertains to a unique graph. Let \mathcal{B}_1 be a D colored graph and select two vertices $v_1, \bar{v}_1 \in \mathcal{B}_1$. The contraction of \mathcal{B}_1 with the pair of vertices, denoted $\mathcal{B}_1/(v_1, \bar{v}_1)$ is the graph obtained from \mathcal{B}_1 by deleting the vertices v_1 and \bar{v}_1 and reconnecting the lines touching them pairwise respecting the colorings.

Again this operation can be performed in two steps. Consider the trace invariant associated to \mathcal{B}_1 and connect the two vertices v_1 and \bar{v}_1 by a dashed line of color 0. Again this line will identify all the indices of the tensor associated

to v_1 with the ones of the tensor associated to \bar{v}_1

$$\begin{aligned} \sum_{n_i^{v_1}, \bar{n}_i^{\bar{v}_1}} \delta_{\{\bar{n}^v, \bar{n}^{\bar{v}}\}}^{\mathcal{B}_1} \left(\prod_i \delta_{n_i^{v_1} \bar{n}_i^{\bar{v}_1}} \right) &= \sum_{n_i^{v_1}, \bar{n}_i^{\bar{v}_1}} \left(\prod_{i=1}^D \prod_{l^i=(v, \bar{v}) \in \mathcal{B}_1} \delta_{n_i^v \bar{n}_i^{\bar{v}}} \right) \left(\prod_i \delta_{n_i^{v_1} \bar{n}_i^{\bar{v}_1}} \right) \\ &= \delta_{\{\bar{n}^v, \bar{n}^{\bar{v}}\}}^{\mathcal{B}/(v_1, \bar{v}_1)}, \end{aligned} \quad (3.18)$$

as both v_1 and \bar{v}_1 are vertices in \mathcal{B}_1 . The contraction is represented graphically in figure 4.

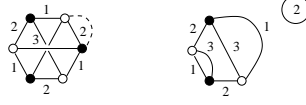


Figure 4. Graphical representation of the contraction of a graph.

The contraction preserves the colorability. Note that the graph $\mathcal{B}/(v_1, \bar{v}_1)$ can potentially be disconnected (see figure 4). We will denote its connected components $[\mathcal{B}/(v_1, \bar{v}_1)]_\rho$. Moreover, some of these connected components can consist in an unique line (as it is the case in figure 4). In this case they are not strictu sensu D colored graphs, but consist in exactly one line (with some color) which closes onto itself. This happens for every line which connects directly v_1 and \bar{v}_1 in \mathcal{B} . Such a line brings a factor N . At the level of the incidence matrix, one builds from $\epsilon_{v\bar{v}}^{\mathcal{B}_1}$ the incidence matrix

$$v \neq v_1, \bar{v} \neq \bar{v}_1 \quad \epsilon_{v\bar{v}}^{\mathcal{B}_1/(v_1, \bar{v}_1)} = \epsilon_{v\bar{v}}^{\mathcal{B}_1} \cup \left(\epsilon_{v_1\bar{v}}^{\mathcal{B}_1} \cap \epsilon_{v\bar{v}_1}^{\mathcal{B}_1} \right). \quad (3.19)$$

and the connected components with no vertices of course do not have an incidence matrix.

These two operations encode the changing of an observable when adding a line of color 0. If the line of color 0 in \mathcal{G} is a tree line (joins \mathcal{B}_1 with some \mathcal{B}_2), then the observables \mathcal{B}_1 and \mathcal{B}_2 are glued. If on the other hand l^0 is a loop line (starts and ends on the same \mathcal{B}_1), then the observable is contracted.

Geometrically these two operations have the following interpretation. Set for now $D = 3$. Then \mathcal{B}_1 and \mathcal{B}_2 represent surfaces. The surface associated to \mathcal{B}_1 is obtained by associating a triangle with edges colored 1, 2 and 3 to each vertex of \mathcal{B}_1 . This induces a coloring of the points (vertices) of the triangle by pairs of colors: 12 is the point common to the edges 1 and 2 and so on. A line in \mathcal{B}_1 represents the unique gluing of the two triangles corresponding to its end vertices which respects *all* the colorings (those of the edges *and* those of the points, see [52] for more details). The gluing $\mathcal{B}_1 \star_{(v_1, \bar{v}_2)} \mathcal{B}_2$ comes to choosing a triangle (corresponding to v_1) on \mathcal{B}_1 , a triangle (corresponding to \bar{v}_2) on \mathcal{B}_2 and gluing the two surfaces along the triangles. This is represented in figure 5 on the left, where we depicted the simplest case of the gluing of two planar surfaces. The contraction is essentially the same thing, just that this time the two triangles belong to the same surface, as represented in figure 5 on the right.



Figure 5. Gluing and contraction of surfaces for tensors of rank $D = 3$.

Note that the topology of the surfaces is changed under these moves (in the example of figure 5 on the right a planar surfaces becomes a genus one surface). This should come as no surprise as the same happens for matrix models: the contraction of a loop leads to two loops. The gluing is just a graphical encoding of surgery on the surfaces. The contraction has a more involved topological interpretation, and it can lead to an increase of the genus (if the two triangles contracted do not share anything), a splitting of the surface into connected components (if the two triangles contracted share at least two vertices but no edges) or no change at all (if the two triangles contracted share edges).

The most important feature of the gluing and the contraction is the following. Consider two observables \mathcal{B}_1 and \mathcal{B}_2 joined by two lines of color 0, (v_1, \bar{v}_2) and (v_2, \bar{v}_1) with $v_1, \bar{v}_1 \in \mathcal{B}_1$ and $v_2, \bar{v}_2 \in \mathcal{B}_2$. The resulting observable can be obtained in two ways, either by gluing along (v_1, \bar{v}_2) and contracting with respect to (v_2, \bar{v}_1) or the reverse. As the end result is unique, we have

$$v_1, \bar{v}_1 \in \mathcal{B}_1, v_2, \bar{v}_2 \in \mathcal{B}_2 \Rightarrow [\mathcal{B}_1 \star_{(v_1, \bar{v}_2)} \mathcal{B}_2] / (v_2, \bar{v}_1) = [\mathcal{B}_2 \star_{(v_2, \bar{v}_1)} \mathcal{B}_1] / (v_1, \bar{v}_2). \quad (3.20)$$

In $D = 2$ dimensions the observables are just bi colored cycles. The gluing of two cycles of lengths p and q always leads to a cycle of length $p + q$ thus the gluing reduces to the addition of the lengths (hence it is associative). The colored gluing of graphs is associative (provided one tracks the vertices at which it is made) only if both expressions $[\mathcal{B} \star_{(v, \bar{v}_1)} \mathcal{B}_1] \star_{(v', \bar{v}_2)} \mathcal{B}_2$ and $\mathcal{B} \star_{(v, \bar{v}_1)} [\mathcal{B}_1 \star_{(v', \bar{v}_2)} \mathcal{B}_2]$ are defined, that is if $v' \in \mathcal{B}_1$. Note however that if $v, v' \in \mathcal{B}$, while $[\mathcal{B} \star_{(v, \bar{v}_1)}] \star_{(v', \bar{v}_2)} \mathcal{B}_2$ is defined, $\mathcal{B} \star_{(v, \bar{v}_1)} [\mathcal{B}_1 \star_{(v', \bar{v}_2)} \mathcal{B}_2]$ is *not*. The gluing is the appropriate generalization of this addition to the D -colored graphs representing the observables of tensor models. In the spirit of the matrix model nomenclature, one should call the trace invariants observables “bubble observables”, and the SDE’s we derive in the next section “bubble equations”.

B. Schwinger Dyson equations and the algebra of constraints

We now derive the SDE’s of tensor models at all orders in N . We subsequently translate them into constraints satisfied by the partition function. The constraints form a Lie algebra, generalizing to all orders in N the D -ary tree algebra identified in [58].

Consider a D -colored graph \mathcal{B}_1 . We chose a vertex $\bar{v}_1 \in \mathcal{B}$ (and mark it). For any \mathcal{B}_1 and $\bar{v}_1 \in \mathcal{B}_1$ the following trivial identity holds

$$\sum_{\vec{p}; n, \bar{n}} \int [dT d\bar{T}] \frac{\delta}{\delta T \bar{T}} \left(\delta_{\vec{n} \bar{v}_1 \vec{p}} \prod_{v \in \mathcal{B}} T_{\vec{n}_v} \prod_{\bar{v} \in \mathcal{B}_1} \bar{T}_{\vec{n}_{\bar{v}}} \delta_{n \bar{n}}^{\mathcal{B}_1} e^{-N^{D-1} S} \right) = 0 \Rightarrow$$

$$\left\langle \sum_{v_1 \in \mathcal{B}_1} \text{Tr}_{\mathcal{B}_1/(v_1, \bar{v}_1)}(T, \bar{T}) \right\rangle - N^{D-1} \sum_{\mathcal{B}} t_{\mathcal{B}} \sum_{v \in \mathcal{B}} \left\langle \sum \text{Tr}_{\mathcal{B} \star_{(v, \bar{v}_1)} \mathcal{B}_1}(T, \bar{T}) \right\rangle = 0. \quad (3.21)$$

We denote the $|\rho|$ connected components of $\mathcal{B}_1/(v_1, \bar{v}_1)$ by $[\mathcal{B}_1/(v_1, \bar{v}_1)]_{\rho}$, with $1 \leq |\rho| \leq D$. If one of these connected components consist in a single line the associated trace is N . Thus

$$\text{Tr}_{\mathcal{B}_1/(v_1, \bar{v}_1)}(T, \bar{T}) = \prod_{\rho=1}^{|\rho|} \text{Tr}_{[\mathcal{B}_1/(v_1, \bar{v}_1)]_{\rho}}(T, \bar{T}). \quad (3.22)$$

The SDE’s translates into a set of differential operators acting on Z indexed by the observable \mathcal{B}_1 and the marked vertex \bar{v}_1

$$\mathcal{L}_{(\mathcal{B}_1, \bar{v}_1)} Z = 0,$$

$$\mathcal{L}_{(\mathcal{B}_1, \bar{v}_1)} = \sum_{v_1 \in \mathcal{B}_1} \prod_{\rho=1}^{|\rho|} \left(-\frac{1}{N^{D-1}} \frac{\partial}{\partial t_{[\mathcal{B}_1/(v_1, \bar{v}_1)]_{\rho}}} \right) + \sum_{\mathcal{B}} t_{\mathcal{B}} \sum_{v \in \mathcal{B}} \frac{\partial}{\partial t_{\mathcal{B} \star_{(v, \bar{v}_1)} \mathcal{B}_1}}, \quad (3.23)$$

and by convention the partial derivative is $-N^D$ if $[\mathcal{B}_1/(v_1, \bar{v}_1)]_{\rho}$ is formed by an unique line. The natural domain of the differential operators $\mathcal{L}_{(\mathcal{B}_1, \bar{v}_1)}$ is the set of invariant functions depending on the coupling constants $t_{\mathcal{B}}$. The constraints form a Lie algebra

Theorem 1. *The commutator of two differential operators $\mathcal{L}_{(\mathcal{B}_1, \bar{v}_1)}$ and $\mathcal{L}_{(\mathcal{B}_2, \bar{v}_2)}$ is*

$$[\mathcal{L}_{(\mathcal{B}_1, \bar{v}_1)}, \mathcal{L}_{(\mathcal{B}_2, \bar{v}_2)}] = \sum_{v \in \mathcal{B}_1} \mathcal{L}_{(\mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2, \bar{v}_1)} - \sum_{v \in \mathcal{B}_2} \mathcal{L}_{(\mathcal{B}_2 \star_{(v, \bar{v}_1)} \mathcal{B}_1, \bar{v}_2)}. \quad (3.24)$$

Proof: The proof is a straightforward computation. We start from the commutator

$$\begin{aligned} & [\mathcal{L}_{(\mathcal{B}_1, \bar{v}_1)}, \mathcal{L}_{(\mathcal{B}_2, \bar{v}_2)}] \\ &= \left[\sum_{v_1 \in \mathcal{B}_1} \prod_{\rho=1}^{|\rho|} \left(-\frac{1}{N^{D-1}} \frac{\partial}{\partial t_{[\mathcal{B}_1/(v_1, \bar{v}_1)]_{\rho}}} \right), \sum_{\mathcal{B}} t_{\mathcal{B}} \sum_{v \in \mathcal{B}} \frac{\partial}{\partial t_{\mathcal{B} \star_{(v, \bar{v}_2)} \mathcal{B}_2}} \right] \\ & - \left[\sum_{\mathcal{B}} t_{\mathcal{B}} \sum_{v \in \mathcal{B}} \frac{\partial}{\partial t_{\mathcal{B} \star_{(v, \bar{v}_1)} \mathcal{B}_1}}, \sum_{v_2 \in \mathcal{B}_2} \prod_{\rho=1}^{|\rho|} \left(-\frac{1}{N^{D-1}} \frac{\partial}{\partial t_{[\mathcal{B}_2/(v_2, \bar{v}_2)]_{\rho}}} \right) \right] \\ & + \left[\sum_{\mathcal{B}} t_{\mathcal{B}} \sum_{v \in \mathcal{B}} \frac{\partial}{\partial t_{\mathcal{B} \star_{(v, \bar{v}_1)} \mathcal{B}_1}}, \sum_{\mathcal{B}'} t_{\mathcal{B}'} \sum_{v' \in \mathcal{B}'} \frac{\partial}{\partial t_{\mathcal{B}' \star_{(v', \bar{v}_2)} \mathcal{B}_2}} \right]. \end{aligned} \quad (3.25)$$

The first line evaluates to

$$\sum_{v_1 \in \mathcal{B}_1} \sum_{\mu=1}^{|\rho|} \prod_{\rho=1, \rho \neq \mu}^{|\rho|} \left(-\frac{1}{N^{D-1}} \frac{\partial}{\partial t} \frac{\partial}{\partial t \left[\mathcal{B}_1 / (v_1, \bar{v}_1) \right]_\rho} \right) \sum_{v \in \left[\mathcal{B}_1 / (v_1, \bar{v}_1) \right]_\mu} \left(-\frac{1}{N^{D-1}} \right) \frac{\partial}{\partial t \left[\mathcal{B}_1 / (v_1, \bar{v}_1) \right]_\mu \star_{(v, \bar{v}_2)} \mathcal{B}_2} . \quad (3.26)$$

Consider the bubble $\mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2$. When reducing with respect to (v_1, \bar{v}_1) with v_1 in \mathcal{B}_1 it will disconnect into several connected components $\left\{ \left[\mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2 \right] / (v_1, \bar{v}_1) \right\}_\rho$. All save one (the one to which v belongs) coincide with the connected components $\left[\mathcal{B}_1 / (v_1, \bar{v}_1) \right]_\rho$. The special one is $\left\{ \left[\mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2 \right] / (v_1, \bar{v}_1) \right\}_\mu = \left[\mathcal{B}_1 / (v_1, \bar{v}_1) \right]_\mu \star_{(v, \bar{v}_2)} \mathcal{B}_2$. Also, $\sum_{\mu=1}^{|\rho|} \sum_{v \in \left[\mathcal{B}_1 / (v_1, \bar{v}_1) \right]_\mu} = \sum_{v \in \mathcal{B}_1, v \neq v_1}$ hence the first line is

$$\sum_{v_1 \in \mathcal{B}_1} \sum_{v \in \mathcal{B}_1, v \neq v_1} \prod_{\rho=1}^{|\rho|} \left(-\frac{1}{N^{D-1}} \frac{\partial}{\partial t} \frac{\partial}{\partial t \left\{ \left[\mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2 \right] / (v_1, \bar{v}_1) \right\}_\rho} \right) , \quad (3.27)$$

and exchanging the sums over v and v_1 it becomes

$$\begin{aligned} & \sum_{v \in \mathcal{B}_1} \sum_{v_1 \in \mathcal{B}_1, v_1 \neq v} \prod_{\rho=1}^{|\rho|} \left(-\frac{1}{N^{D-1}} \frac{\partial}{\partial t} \frac{\partial}{\partial t \left\{ \left[\mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2 \right] / (v_1, \bar{v}_1) \right\}_\rho} \right) \\ &= \sum_{v \in \mathcal{B}_1} \sum_{v' \in \mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2} \prod_{\rho=1}^{|\rho|} \left(-\frac{1}{N^{D-1}} \frac{\partial}{\partial t} \frac{\partial}{\partial t \left\{ \left[\mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2 \right] / (v', \bar{v}_1) \right\}_\rho} \right) \\ & \quad - \sum_{v \in \mathcal{B}_1} \sum_{v' \in \mathcal{B}_2} \prod_{\rho=1}^{|\rho|} \left(-\frac{1}{N^{D-1}} \frac{\partial}{\partial t} \frac{\partial}{\partial t \left\{ \left[\mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2 \right] / (v', \bar{v}_1) \right\}_\rho} \right) \end{aligned} \quad (3.28)$$

where we relabeled v_1 by v' and we added and subtracted the terms with $v' \in \mathcal{B}_2$. Recall that by equation (3.20), if $v, \bar{v}_1 \in \mathcal{B}_1$ and $v', \bar{v}_2 \in \mathcal{B}_2$ then

$$\left[\mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2 \right] / (v', \bar{v}_1) = \left[\mathcal{B}_2 \star_{(v', \bar{v}_1)} \mathcal{B}_1 \right] / (v, \bar{v}_2) , \quad (3.29)$$

hence the first two lines in eq. (3.25) yield

$$\begin{aligned} & \sum_{v \in \mathcal{B}_1} \sum_{v' \in \mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2} \prod_{\rho=1}^{|\rho|} \left(-\frac{1}{N^{D-1}} \frac{\partial}{\partial t} \frac{\partial}{\partial t \left\{ \left[\mathcal{B}_1 \star_{(v, \bar{v}_2)} \mathcal{B}_2 \right] / (v', \bar{v}_1) \right\}_\rho} \right) \\ & - \sum_{v \in \mathcal{B}_2} \sum_{v' \in \mathcal{B}_2 \star_{(v, \bar{v}_1)} \mathcal{B}_1} \prod_{\rho=1}^{|\rho|} \left(-\frac{1}{N^{D-1}} \frac{\partial}{\partial t} \frac{\partial}{\partial t \left\{ \left[\mathcal{B}_2 \star_{(v, \bar{v}_1)} \mathcal{B}_1 \right] / (v', \bar{v}_2) \right\}_\rho} \right) . \end{aligned} \quad (3.30)$$

The third line in equation (3.25) writes

$$\sum_{\mathcal{B}} t_{\mathcal{B}} \sum_{v \in \mathcal{B}} \sum_{v' \in \mathcal{B} \star_{(v, \bar{v}_1)} \mathcal{B}_1} \frac{\partial}{\partial t \left[\mathcal{B} \star_{(v, \bar{v}_1)} \mathcal{B}_1 \right] \star_{(v', \bar{v}_2)} \mathcal{B}_2} - (1 \leftrightarrow 2) . \quad (3.31)$$

We separate the terms with $v' \in \mathcal{B}_1$ from the terms with $v' \in \mathcal{B}$ to get

$$\begin{aligned} &= \sum_{\mathcal{B}} t_{\mathcal{B}} \sum_{v \in \mathcal{B}} \sum_{v' \in \mathcal{B}_1} \frac{\partial}{\partial t \left[\mathcal{B} \star_{(v, \bar{v}_1)} \mathcal{B}_1 \right] \star_{(v', \bar{v}_2)} \mathcal{B}_2} + \sum_{\mathcal{B}} t_{\mathcal{B}} \sum_{v \in \mathcal{B}} \sum_{v' \in \mathcal{B}, v' \neq v} \frac{\partial}{\partial t \left[\mathcal{B} \star_{(v, \bar{v}_1)} \mathcal{B}_1 \right] \star_{(v', \bar{v}_2)} \mathcal{B}_2} - (1 \leftrightarrow 2) \\ &= \sum_{v' \in \mathcal{B}_1} \sum_{\mathcal{B}} t_{\mathcal{B}} \sum_{v \in \mathcal{B}} \frac{\partial}{\partial t \left[\mathcal{B} \star_{(v, \bar{v}_1)} \left[\mathcal{B}_1 \star_{(v', \bar{v}_2)} \mathcal{B}_2 \right] \right]} - (1 \leftrightarrow 2) \end{aligned} \quad (3.32)$$

Adding equations (3.30) with (3.31) we obtain, relabeling some dummy indices

$$\begin{aligned} & \left[\mathcal{L}_{(\mathcal{B}_1, \bar{v}_1)}, \mathcal{L}_{(\mathcal{B}_2, \bar{v}_2)} \right] = \\ & \sum_{w \in \mathcal{B}_1} \left\{ \sum_{v' \in \mathcal{B}_1 \star_{(w, \bar{v}_2)} \mathcal{B}_2} \prod_{\rho=1}^{|\rho|} \left(-\frac{1}{N^{D-1}} \frac{\partial}{\partial t} \frac{\partial}{\left\{ \left[\mathcal{B}_1 \star_{(w, \bar{v}_2)} \mathcal{B}_2 \right] / (v', \bar{v}_1) \right\}_\rho} \right) - \sum_{\mathcal{B}} t_{\mathcal{B}} \sum_{v \in \mathcal{B}} \frac{\partial}{\partial t} \frac{\partial}{\mathcal{B} \star_{(v, \bar{v}_1)} \left[\mathcal{B}_1 \star_{(w, \bar{v}_2)} \mathcal{B}_2 \right]} \right\} - (1 \leftrightarrow 2), \end{aligned} \quad (3.33)$$

hence

$$\left[\mathcal{L}_{(\mathcal{B}_1, \bar{v}_1)}, \mathcal{L}_{(\mathcal{B}_2, \bar{v}_2)} \right] = \sum_{w \in \mathcal{B}_1} \mathcal{L}_{(\mathcal{B}_1 \star_{(w, \bar{v}_2)} \mathcal{B}_2, \bar{v}_1)} - \sum_{w \in \mathcal{B}_2} \mathcal{L}_{(\mathcal{B}_2 \star_{(w, \bar{v}_1)} \mathcal{B}_1, \bar{v}_2)}. \quad (3.34)$$

□

This Lie algebra admits a closed Lie subalgebra. The leading order observables, the melons [38], are indexed by colored rooted D -ary trees \mathcal{T} . It is easy to check that gluing of the observables $\mathcal{B}_1 \star_{(v_1, \bar{v}_2)} \mathcal{B}_2$ reproduces the gluing of their associated trees $\mathcal{T}_1 \star_V \mathcal{T}_2$, as defined in [58]. The melonic observables are closed under this composition (as the gluing of trees leads to trees), hence the algebra indexed by D -ary trees identified in [58] is a Lie subalgebra of the full constraints algebra.

IV. CONCLUSION

We have derived in this paper the SDE's of tensor models for a generic complex tensor at all orders in $1/N$. They translate into a Lie algebra of constraints obeyed by the partition function. The algebra is indexed by colored graphs and generalizes to all orders in $1/N$ the algebra indexed by D -ary trees of the leading order observables. The algebra indexed by D -ary trees closes a Lie subalgebra of the full constraints algebra.

The study of this algebra of constraints, primarily of its central extensions and unitary representations, is a prerequisite for the full classification of the continuum limits of tensor models. The study of its representations would benefit from identifying various Lie subalgebras, and studying the their representations first. We already possess a candidate, the leading order algebra indexed by D -ary trees. Other, simpler subalgebras can readily be identified: for instance the Virasoro algebra itself is a subalgebra of the full constraints algebra [50] (in fact, following the results of [50], one can identify several distinct copies of the Virasoro algebra as subalgebras of the full constraints algebra). The continuum SDE's should be understood in some appropriate double scaling limit and the continuum operators should be identified [18, 20]. Other aspects of the emergent continuous geometry like its effective spectral and Hausdorff dimensions must also be analyzed. Analytic control of the continuum limit is a prerequisite in order to use the random tensor models to investigate conformal field theories, statistical models in random geometry and quantum gravity in arbitrary dimensions.

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